Differential Equations&*Mathematica*

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DE.01 Transition from Calculus to DiffEq: The Exponential Differential Equation $y'[t] + r\,y[t] = f[t]$ *BASICS*

Mathematica Initializations

B.3) Steady state for $y'[x] + r\,y[x] = f[t]$ with $r > 0$: Any solution eventually settles into the same steady state. If $r < 0$, then all bets are off

□ B.3.a.i) Merging plots when r > 0

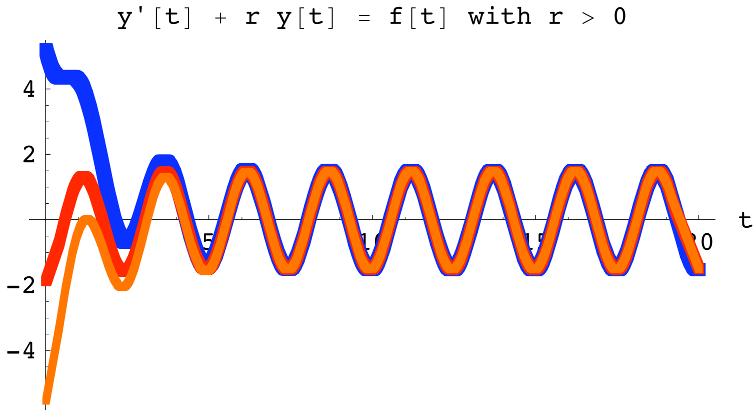
Here are three plots of solutions a random forced exponential diffeq

$y'[t] + r\,y[t] = 4\,\mathrm{Sin}[2.5\,t]$, (with $r > 0$)

with random starting values on y[0]:

```
r = Random[Real, {0.5, 1.0}];
f[t_] = 4 Sin[2.5 t];
endtime = 20;

Clear[y, y1, y2, y3, t];
starter1 = Random[Real, {5, 10}]; starter2 = Random[Real, {-2, 2}];
starter3 = Random[Real, {-10, -5}]; y1[t_] = E^(-r t) starter1 + E^(-r t) ∫_0^t E^(r s) f[s] ds;
y2[t_] = E^(-r t) starter2 + E^(-r t) ∫_0^t E^(r s) f[s] ds; y3[t_] = E^(-r t) starter3 + E^(-r t) ∫_0^t E^(r s) f[s] ds;
plots1 = Plot[{y1[t], y2[t], y3[t]}, {t, 0, endtime},
  PlotStyle → {{Thickness[0.018], Blue}, {Thickness[0.014], Red},
    {Thickness[0.01], CadmiumOrange}}, PlotRange → All, AspectRatio → Automatic,
  AxesLabel → {"t", ""}, PlotLabel → "y'[t] + r y[t] = f[t] with r > 0"];
```



Rerun several times.

Describe what you see and explain why you see it.

Answer:

Take a look at another one – this time it will be

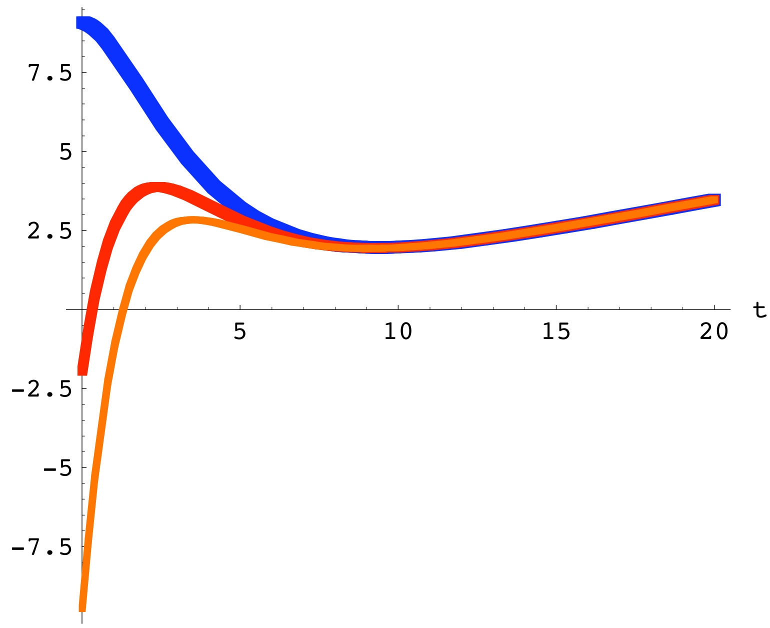
$$y'[t] + r\,y[t] = 6\,E^{-0.4\,t} + p\,t$$

with random r > 0, random choice of p, and random starting values on y[0]:

```
r = Random[Real, {0.5, 1.0}];
p = Random[Real, {-0.5, 0.5}]; f[t_] = 6 E^(-0.4 t) + p t;
endtime = 20;

Clear[y, y1, y2, y3, t];
y'[t] + r y[t] == f[t]
starter1 = Random[Real, {5, 10}]; starter2 = Random[Real, {-2, 2}];
starter3 = Random[Real, {-10, -5}]; y1[t_] = E^(-r t) starter1 + E^(-r t) ∫_0^t E^(r s) f[s] ds;
y2[t_] = E^(-r t) starter2 + E^(-r t) ∫_0^t E^(r s) f[s] ds;
y3[t_] = E^(-r t) starter3 + E^(-r t) ∫_0^t E^(r s) f[s] ds; plots2 =
 Plot[{y1[t], y2[t], y3[t]}, {t, 0, endtime}, PlotStyle → {{Thickness[0.018], Blue},
    {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange → All, AspectRatio → Automatic, AxesLabel → {"t", ""}];
```

$0.673667\,y[t] + y'[t] == 6\,e^{-0.4\,t} + 0.126453\,t$



Rerun several times.

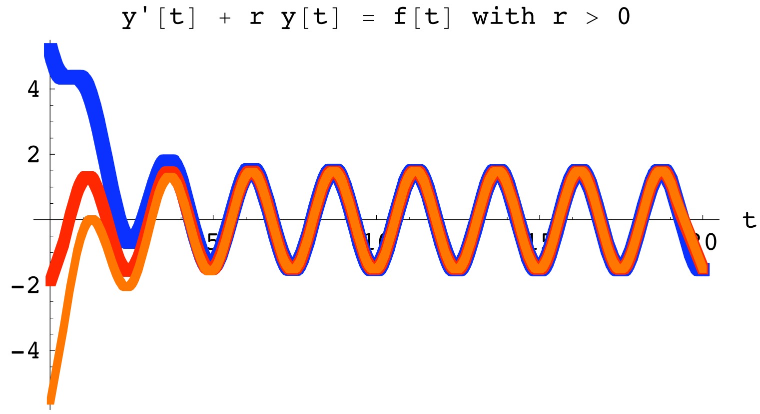
Every time the three solutions begin their trip in totally different ways, but when t is large, you need a scorecard to tell them apart.
In other words, they all settle into the same steady state behavior-regardless of the starting values on y[0]. The long term behavior of the solutions is totally insensitive to the starting value on y[0].

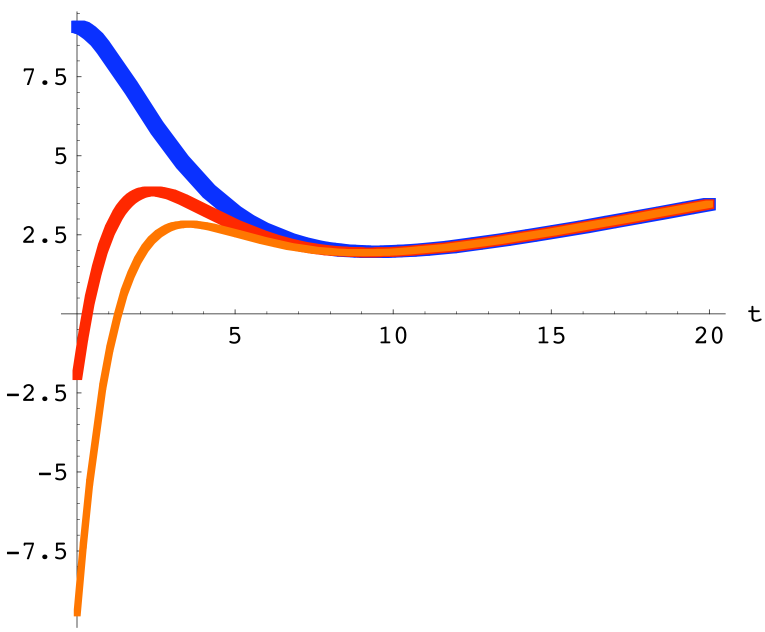
To explain why this happens, look at the formula

$$y[t] = E^{-r\,t}\,\text{starter} + E^{-r\,t} \int_0^t E^{r\,s}\, f[s]\, ds$$

If $r > 0$, then as t gets large, you are guaranteed that $E^{-r\,t}$ starter $\to 0$.
The upshot: In the long run, $E^{-r\,t} \int_0^t E^{r\,s}\, f[s]\, ds$
That's what you're seeing here:

```
Show[plots1];
Show[plots2];
```





□ **B.3.a.ii) Steady state (long term) behavior**

Does this mean that when you have a a forced exponential diffeq

$y'[t] + r\,y[t] = f[t]$, with $r > 0$

then the long term steady state behavior of solutions

→ depends only on the behavior of f[t]

and

→ does not depend not on the starting value for y[0]?

□ **Answer:**

Yes!!!!

□ B.3.a.ii) When r < 0, then all bets are off

What happens when r < 0?

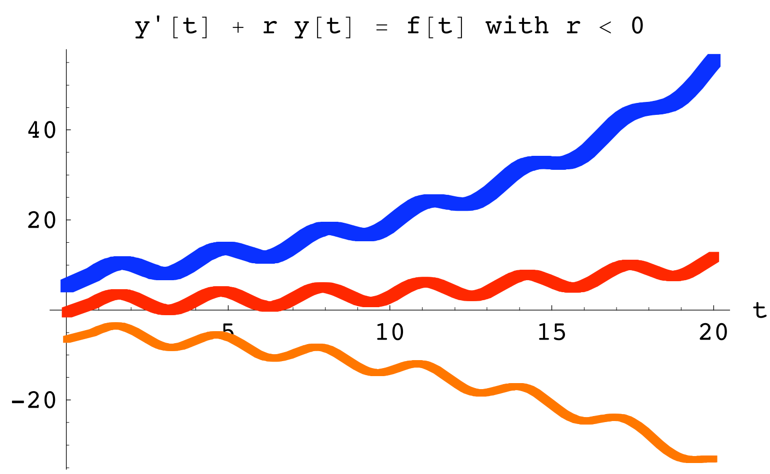
□ Answer:

Try it and see:

```
r = -0.1;
f[t_] = 3.8 Sin[2 t];
endtime = 20;

Clear[y, y1, y2, y3, t];
y'[t] + r y[t] == f[t]
starter1 = Random[Real, {5, 10}]; starter2 = Random[Real, {-2, 2}];
starter3 = Random[Real, {-10, -5}]; y1[t_] = E^(-r t) starter1 + E^(-r t) ∫_0^t E^(r s) f[s] ds;
y2[t_] = E^(-r t) starter2 + E^(-r t) ∫_0^t E^(r s) f[s] ds; y3[t_] = E^(-r t) starter3 + E^(-r t) ∫_0^t E^(r s) f[s] ds;
plots = Plot[{y1[t], y2[t], y3[t]}, {t, 0, endtime},
  PlotStyle → {{Thickness[0.018], Blue}, {Thickness[0.014], Red},
    {Thickness[0.01], CadmiumOrange}}, PlotRange → All, AspectRatio → 1/GoldenRatio,
  AxesLabel → {"t", ""}, PlotLabel → "y'[t] + r y[t] = f[t] with r < 0"];
```

```
-0.1 y[t] + y'[t] == 3.8 Sin[2 t]
```



If r < 0, the solutions don't merge.

To see why, look at the formula

$$y[t] = E^{-r t}\,\text{starter} + E^{-r t}\int_0^t E^{r s}\, f[s]\, ds.$$

When r < 0, the term $E^{-r t}$ starter does not go away as t gets big.

In fact $E^{-r t}$ starter gets bigger and bigger as t gets large.

That's why different starter values on y[0] yield increasingly different plots as t gets large.